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On the wake in the low-Reynolds-number flow behind an impulsively started circular cylinder

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Summary

This analysis completes the authors' singular perturbations type of solution of the title problem, when the cylinder acquires instantaneously a uniform velocity. As reported the solution consists of three expansions which represent the flow in three different space-time subdomains. It is shown here that there exists a fourth subdomain. An appropriate additional expansion is developed and matched with the other three. This latter expansion represents the flow late in the process in a wake region. This wake extends all the way downstream to infinity. Its width is comparable to the diameter of the obstacle.

1. Introduction

As presented by the authors in a previous paper [1] their solution for the flow due to an impulsively started cylinder is incomplete. It is expressed in terms of three expansions. One which holds "early" in the process "throughout" the exterior of the cylinder: (e, t). Two which hold late in the process: a "late-inner" (l, i) and a "late-outer" (l, o). However although it was tacitly assumed that the latter expansion represents the flow everywhere beyond the immediate vicinity of the obstacle, it was subsequently found that this assumption is incorrect. Evidently the (l, o) representation of the stream function has a discontinuous derivative across the half plane $(X > 0)$ behind the moving cylinder.

This discontinuity is barely perceptible. It is in the second derivative while the stream function itself as well as the first and third derivatives are continuous. However this discontinuity is significant because it is an inherent feature of the late-outer flow field due to a cylinder acquiring a uniform velocity instantaneously. In other words, it appears that one cannot construct a solution of the unsteady fourth-order Oseen equation, which prevails in the (l, o) subdomain, that satisfies all the appropriate conditions. By that we mean that it matches (e, t) and (l, i) expansions, meets the regularity conditions at infinity and has its zero to third derivative continuous everywhere in the outer field.

This finding raises the question whether the three-expansions solution as proposed in [1] is valid. The authors show here that it is in the sense that a potential solution is a valid representation of a high-Reynolds-number flow past an obstacle. Thus, just as the irrotational solution holds throughout the exterior of the obstacle but not on its surface, the three-expansions solution proposed in [1] holds almost but not quite throughout the

(l, o) space-time subdomain under discussion. Furthermore just as the inadequacy of the classical irrotational flow can be mended by appending a boundary layer, the discontinuity of the "late-outer" expansion along the half-plane behind the cylinder can be mended by constructing an additional, fourth, expansion. It expresses the flow over this plane and its immediate vicinity late in the process. Adhering to the notation of [1] the newly discovered subdomain is designated "late-wake" and this is abbreviated as (l, w).

2. The incomplete solution

To explain the construction of the additional fourth expansion the development of the first three will be briefly recapitulated. Together all four expansions constitute an approximate solution of the time-dependent vorticity equation

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \mathrm{Re}\frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} = \nabla^4\psi. \tag{1}$$

Here ψ is the stream function normalized with respect to the characteristic velocity U and the radius of the cylinder a. The space and time variables (x, y, t) are scaled with respect to a and the kinematic viscosity ν. The Reynolds number Re is aU/ν and it is assumed to be small.

This scaling of the independent variables characterizes the flow early in the process throughout. Thus for the case of a cylinder acquiring instantenously a uniform velocity U the solution in the (e, t) subdomain is obtained by perturbing Eqn. (1). But late in the process the appropriate time scale is ν/U^2 so that the time variable is $T = t\mathrm{Re}^2$. The characteristic length scales are a and ν/U close to cylinder and away from it. Hence the space variables in the (l, i) and (l, o) subdomains are r and $R = r\mathrm{Re}$, respectively. It follows that in the latter two subdomains the governing equation takes different forms. Accordingly the solution for ψ is differently approximated in each regime.

As explained, the authors' main concern is the nature of the solution in the "late-outer" subdomain. Following Proudman and Pearson [2] it was assumed that the appropriate expansion there is

$$\psi^{(l,o)} \sim \frac{1}{\mathrm{Re}} YH(T) + \frac{\Delta}{\mathrm{Re}}\psi_1^{(l,o)}. \tag{2}$$

Since $H(t)$ stands for Heaviside's step function the leading term represents the instantaneously started uniform flow past the stationary cylinder (if the co-ordinates are fixed to the cylinder departing from rest in otherwise quiescent fluid). The second term is assumed to be of $O(\Delta/\mathrm{Re})$ where $\Delta(\mathrm{Re})$ is $[\ln(\mathrm{Re}^{-1}) + k]^{-1}$ and k is the constant introduced by Kaplun [3]. Thus $\psi^{(l,o)}$ satisfies the unsteady Oseen equation

$$\left(\frac{\partial}{\partial T} + \frac{\partial}{\partial X} - \overline{\nabla}^2\right)\overline{\nabla}^2\psi_1^{(l,o)} = 0. \tag{3}$$

Its timewise Laplace-S-transform, $\Psi^{(l,o)}$, that is regular at infinity and matches the expansions prevailing in the (l, i) and (e, t) subdomains, is given in Eqn. (24) of [1] in the form of an integral over the negative X-axis. The discontinuity associated with $\Psi^{(l,o)}$, and its other properties are more easily observed by writing that equation as follows:

$$\Psi_1^{(l,o)} = -\frac{2}{S}\frac{\partial}{\partial Y}\int_{-\infty}^{x} \exp(-Sx)\left[e^{(S+1/2)\eta}K_0(\zeta\bar{R}/2) + e^{S\eta}\ln\bar{R}\right]d\eta \tag{4}$$

where

$$\bar{R}^2 = \eta^2 + Y^2, \qquad \zeta = (4S+1)^{1/2}. \tag{5}$$

Note that the modified Bessel function of the second kind, which appears in the integral, is logarithmically singular for vanishing $\bar{R}$. But this singularity as well as that associated with the term $\ln\bar{R}$ are of no consequence unless both Y and η vanish. This occurs only when the formula is used to calculate $\Psi_1^{(l,o)}$, along $0 \leqslant X < \infty$ and $Y = 0$. It follows that Eqn. (4) holds everywhere except along the positive X-axis. Moreover, in that domain $\Psi_1^{(l,o)}$ can be differentiated any number of times with respect to X and Y. But to calculate $\Psi_1^{(l,o)}$ and its derivative along the positive X-axis one must adopt a limiting process. That is, one must calculate the integral of Eqn. (4) for $Y = \pm\epsilon$ then let ϵ approach zero. It can be easily shown that when $\Psi_1^{(l,o)}$ itself is calculated in that manner, both singularities behave like $Y(\eta^2 + Y^2)^{-1}$. Since they have different signs, their contributions to both limits cancel out and one gets the result

$$\Psi_1^{(l,o)}(X, +0) = \Psi_1^{(l,o)}(X, -0). \tag{6}$$

On the other hand the limiting values of the second derivative are not the same whence the following discontinuity

$$\frac{\partial^2\Psi_1^{(l,o)}}{\partial Y^2}(X, +0) = -\frac{\partial^2\Psi^{(l,o)}}{\partial Y^2}(X, -0) = -\tfrac{9}{16}\pi\zeta^2. \tag{7}$$

It is easily seen that the odd derivatives are even in Y and they are continuous across $0 < X < \infty$.

The discontinuity in the second derivative is irreconcilable with the assumption that the unsteady fourth-order Oseen equation (3) holds everywhere beyond the immediate vicinity of the obstacle. There is no way to "erase" it by adding an appropriate expression to the right-hand side of Eqn. (4). It is also of interest to note that it is not only in case (i) of [1] that the solution of the Oseen equation fails along the positive axis, but also in case (ii) where the cylinder undergoes a uniform acceleration.

To demonstrate the failure in case (ii) we show that the solution for the late-outer-disturbance stream function as given in Eqn. (40) of [1] is not valid on the positive X-axis. It is well known that the asymptotic behaviour of a Fourier integral (40) for $Y \to 0$ is determined by large values of m (see Lighthill's text [4]). Thus, for large values of m we may employ the asymptotic behaviour of the Airy functions [2] to express the kernel of (40) as follows

$$\Lambda(\xi, S, m, \beta) \sim \frac{2^{-1/3}}{m}e^{-m|\xi|}Ai\left(\frac{\xi - 2S + \beta}{2^{1/3}}\right). \tag{8}$$

Then, integrating with respect to m, one gets

$$\frac{\partial \Psi_1^{(l,o)}}{\partial Y}(X, 0, S) = -\tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\beta F(\beta)\left\{\int_{-\infty}^{x} \frac{Ai\left(\frac{\xi - 2S + \beta}{2^{1/3}}\right)}{|\xi| + (X - \xi)}\mathrm{d}\xi + \int_{x}^{\infty} \frac{Ai\left(\frac{\xi - 2S + \beta}{2^{1/3}}\right)}{|\xi| + (\xi - X)}\mathrm{d}\xi\right\} \tag{9}$$

and this expression may be further reduced to

$$\frac{\partial \Psi_1^{(l,o)}}{\partial Y}(X, 0, S) = -\tfrac{3}{2}\int_{-\infty}^{x} \frac{\mathrm{d}\xi}{(S - \xi/2)^2[|\xi| + (X - \xi)]} - \tfrac{3}{2}\int_{x}^{\infty} \frac{\mathrm{d}\xi}{(S - \xi/2)^2[|\xi| + (\xi - X)]}. \tag{10}$$

Then, using relationship C.1 of [1] which reads

$$\int_{-\infty}^{\infty} F(\beta) Ai\left(\frac{\beta - 2S}{2^{1/3}}\right)\mathrm{d}\beta = 3/S^2, \tag{11}$$

and interpreting the integrals in Lighthill's generalized-function sense, we get

$$\frac{\partial \Psi_1^{(l,o)}}{\partial Y}(X, 0, S) = \frac{12}{(4S - X)^2}[\log|2S - X| - \log 2S]. \tag{12}$$

Evidently the inverse Laplace transform of this expression behaves like Te^{XT}. It follows that in case (ii) $\Psi_1^{(l,o)}$ cannot be defined over the entire positive X-axis.

Clearly both the discontinuity of $\Psi_1^{(l,o)}$ in case (i) and the irregular behaviour of the corresponding component in case (ii) occur over the semi-infinite plane downstream of the cylinders' axis. None of these difficulties would emerge had one assumed that late in the process this plane and its immediate vicinity are not part of the (l, o) subdomain. The authors assume just that, and in the next section a solution for such fourth subdomain for case (i) is developed. The fact that it matches the three other expansions falls short of a proof but enhances one's belief that as time progresses a recognizable trailing-wake region develops behind the obstacle departing from rest.

3. The late wake expansion

As explained the additional expansion is assumed to hold over the plane $0 < X < \infty$, $Y = 0$ and its immediate vicinity late in the process under discussion. The lateral width of this fourth subdomain is taken to be finite but much smaller than ν/U, the length scale characterizing the (l, o) subdomain in case (i). Thus in terms of Y, $\partial^2\Psi/\partial Y^2$ appears to be discontinuous across a region of zero thickness.

To be precise the thickness of the wake region is assumed comparable to the diameter of the cylinder. The physical implication of this assumption is that the wake is like a shadow cast by an obstacle placed in a uniform parallel stream. It follows from this assumption that space variations across the wake are characterized by a. On the other hand, since that region is far from the obstacle, the length scale characterizing the axial space co-ordinate and the time co-ordinate scaling must be independent of a. Therefore the "late-wake" expansion $\psi^{(l,w)}$ is a function of (X, y, T). This will be verified by successfully matching it with all three other expansions. Together all four constitute the solution of the title problem.

In fact, assuming that the expansions match, we obtain the form of solution for $\psi^{(l,w)}$ by recasting the Laplace transform of $\psi^{(l,o)}$, $\Psi^{(l,o)}$, in terms of (X, y, S) as follows

$$\Psi^{(l,o)}(X, R_e y, S) = \frac{1}{S} y + \frac{\Delta}{R_e}\left(\sum_{n=0}^{\infty} \frac{\partial^n \Psi_1^{(l,o)}}{\partial Y^n}(X, 0\pm, S) R_e^n y^n\right), \qquad y \gtrless 0. \tag{13}$$

Noting that $\Psi_1^{(l,o)}$ vanishes on the negative X-axis while the first derivative does not,

$$\psi^{(l,w)} \sim H(T) y + \Delta\psi_1^{(l,w)}(X, y, T). \tag{14}$$

Thus the troublesome second derivative is of no consequence when, at most, two terms are retained in each expansion. Nor does it have any influence on a more refined solution. One can show that in such solution the late-inner expansion is in the form of the powers Δ^n, while in late-outer expansion additional terms are of $O(\Delta^n/\mathrm{Re})$. Matching considerations therefore indicate that additional terms in expansion $\psi^{(l,w)}$ are also of $O(\Delta^n)$. These are bigger than the contribution of the second derivative which is of $O(\Delta \mathrm{Re})$. The latter is therefore negligible no matter to what extent the present solution is refined.

Evidently the solution for the Laplace transform of $\psi^{(l,w)}$ is

$$\psi_1^{(l,w)} = y\frac{2}{S}\exp(-SX)\int_{-\infty}^{x}\left[\exp((S+1/2)\xi)\left(\frac{\zeta}{2\xi}\right)K_1(\zeta\xi/2) - \exp(S\xi)/\xi^2\right]\mathrm{d}\xi. \tag{15}$$

Note that by recasting the governing equation in terms of (X, y, T) one gets

$$\frac{\partial^4}{\partial y^4}\left(y + \Delta\psi_1^{(l,w)}\right) = 0 \tag{16}$$

which is satisfied. Matching with the late-outer expansion has already been shown while the matching with the (l, i) and (e, t) solution is immediate.

4. Discussion

The term "wake" means an imprint cast by an obstacle placed in a stream. In the case at hand the imprint that develops as time progresses is in the form of recognizable regime. In it the flow is basically axial. Indeed, the results (14) and (15) suggest that the ratio between the velocity components $(-\partial\psi/\partial x)/(\partial\psi/\partial y)$ is of $O(\Delta \mathrm{Re})$, which is small. Note that at every instant the axial component $\partial\psi/\partial y$ is uniform across the wake and its magnitude is obtained by setting $Y=0$ over $0<X<\infty$ in the late-outer solution. Because of the fluids' incompressibility axial variations in that component are compensated by transverse motion, but it is of much smaller magnitude.

A qualitatively similar situation prevails in the wake trailing a two-dimensional obscale placed in a steady stream when the Reynolds number is high. Again that region is distinct and the flow in it is basically axial. But there are some quantitative differences. In the high-Reynolds-number wake the ratio between the velocity components, as defined, is of $O(\mathrm{Re}^{-1/2})$. This feature is associated with the scaling adopted when Re is high and it reveals that in such case the wake flow is governed by the boundary-layer equation. Thus while the latter constitutes a balance between the effects of inertia and viscosity, in the case at hand the wake is viscosity dominated.

References

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